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NUMERICAL ANALYSIS OF FLUID FLOW AND MASS TRANSFER IN A CHANNEL WITH A POROUS BOTTOM WALL

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SUMMARY

The flow of a solution between parallel plates is considered. The bottom plate is porous, while the top one is an impermeable solid. A computer program based on the control volume approach was developed to analyse the flow and concentration fields. The effects of the slip at the porous wall on the velocity and particle concentration distributions were investigated. It was observed that as the slip increases, the concentration on the porous wall decreases and the maximum velocity moves towards the porous wall. The concentration on the porous wall increases in the flow direction. This increase in the particle concentration along the porous wall may cause a reduction of the porosity and hence a variation in the suction rate along the porous wall. In order to take this effect into account, a linearly varying transverse velocity along the porous wall was considered. The results were compared with the data available in the literature.

KEY WORDS: porous wall; solute concentration; membrane filtration

1. INTRODUCTION

In recent years, membrane filtration has become an important solid–liquid separation process. This process is employed in water treatment facilities, separation of plasma from blood, etc. In addition to these technological applications, flows over a porous medium are also seen in natural phenomena. Flow of underground water and flow in river beds are two examples.

Membrane filtration processes involve the flow of a solution tangentially over a porous wall (see Figure 1). In general the flow is driven by a pressure differential. As the fluid containing particles flows over the porous membrane, clean solvent (with components smaller than the pores) flows through the membrane. Larger particles accumulate on the membrane surface and form a layer.

Analysis of the problem involves the prediction of the velocity distribution and the distribution of the particle concentration in the solution. In the past, researchers[1–3] used approximate methods for the analysis of the problem. Then, using the predicted flow field, the particle concentration distribution was solved by employing a finite difference technique. Berman[4] derived approximate expressions for the fluid velocity components assuming a sufficiently small wall Reynolds number ($Re_w = hv_w/\nu$) and no slip between the fluid and the porous wall. Beavers and Joseph[1] proved the existence of a non-zero tangential velocity (slip) on the surface of a permeable boundary. This situation poses a difficulty for the determination of the tangential component of the velocity at the interface of the fluid and the porous wall. Various researchers have handled this boundary condition differently. Using the statistical approach to extend Darcy's law to the porous medium, Saffman[5] derived an expression for the slip velocity as

$$u_{\text{slip}} = \frac{\sqrt{k}}{\alpha}\frac{\partial u}{\partial y} + O(k), \tag{1}$$

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where k is the permeability of the porous medium and α is a dimensionless constant which depends on the structure of the porous medium. Employing the above expression as the boundary condition on the porous wall, Chellam *et al.*[3] solved the Navier–Stokes equation using a perturbation technique. They investigated the effect of the slip on the velocity field.

In most of the studies found in the literature a perturbation method was employed. In this technique a streamfunction is defined in the form

$$\psi(x,\ y) = (h\bar{u}_o - v_w x) f(y/h), \tag{2}$$

where f is a dimensionless function to be determined. When the above expression is substituted into the Navier–Stokes equations, an ordinary non-linear differential equation for f is obtained which is similar to the Falkner–Skan equation and solved by a perturbation method. It was shown that normalized axial velocity profiles (u/u_m) are similar to each other at every cross-section along the flow. Here u_m is the local axial velocity at the centreline.

In order to utilize the perturbation methods, the transverse velocity on the porous wall needs to be constant. Otherwise, by using the streamfunction defined in equation (2), the Navier–Stokes equations cannot be reduced to an ordinary differential equation. On the other hand, in practical applications the transverse velocity on the porous wall is not constant along the porous wall owing to the variation in the pressure and the accumulation of particles on the permeable wall. Hence a solution technique taking this feature of the problem into account should be developed.

In the present study the Newtonian, incompressible and viscous laminar flow of a solution between two parallel plates was considered (see Figure 1). The bottom plate is a permeable porous wall, while the top one is an impermeable solid. The flow is driven by a pressure gradient. The steady state two-dimensional Navier–Stokes equations were solved numerically to predict the velocity distribution in the flow field. The slip on the porous wall was taken into account in the computations. To investigate the effect of the slip on the velocity field, computations were performed with various slip coefficients. Computations were also carried out with constant and variable transverse velocities (v_w) on the porous wall. The convection–diffusion equation for particles was solved to obtain the concentration distribution of the particles. The results obtained were compared with the data available in the literature.

2. MATHEMATICAL FORMULATION

Analysis of the problem involves the solution of the equations of fluid motion and the particle concentration equation in a Cartesian co-ordinate system with appropriate boundary conditions.

Fluid flow equations

The geometry of the problem and the co-ordinate system are shown in Figure 1. The length and height of the channel are denoted by L and h respectively. Compared with the height, the width of the channel is very large. Hence the flow is considered to be two-dimensional. Then the steady state Navier–Stokes equations are written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{4}$$

where u and v are the velocity components in the x- and y-direction respectively and P is the pressure.

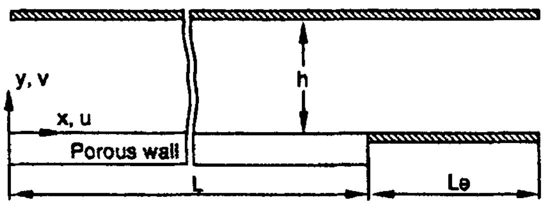

Figure 1. Geometry and co-ordinate system of problem

The continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{5}$$

Equations (3)–(5) are subject to the following boundary conditions.

At the inlet ($x=0$). At the inlet the velocity is assumed to be uniform, i.e.

$$u(0,\ y)=u_{\mathrm{o}}, \tag{6a}$$

$$v(0,\ y)=0. \tag{6b}$$

On the upper wall ($y=h$). For the horizontal component of the velocity vector the no-slip condition is satisfied, i.e.

$$u(x,\ h)=0. \tag{7a}$$

For the vertical component of the velocity vector the impermeability condition is applied, i.e.

$$v(x,\ h)=0. \tag{7b}$$

On the bottom wall ($y=0$). As the solvent flows in the transverse direction in the porous membrane, it also flows in the streamwise direction. Therefore there exists a slip between the fluid and the porous wall. The expression for the slip derived by Saffman[5] is used in the computations:

$$u(x,\ 0)=u_{\mathrm{slip}}=\frac{\sqrt{k}}{\alpha}\left.\frac{\partial u}{\partial y}\right|_{y=0}=\phi_{\mathrm{slip}}\left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{8a}$$

Here $\sqrt{k}/\alpha$ is defined to be ϕ_{slip}, which represents the slip between the fluid and the porous wall.

On the porous wall the fluid also flows in the transverse direction. In general it may be variable along the porous wall. Hence it can be written as

$$v(x,\ 0)=v_{\mathrm{w}}(x). \tag{8b}$$

At the exit ($x=L$). At the exit the values of the flow parameters cannot be prescribed prior to the solution, because they depend on the inlet conditions and the condition on the porous wall. Essentially, the conditions at the exit are the outcome of the computations. To overcome this difficulty, the computational domain was extended beyond the end of the porous wall (beyond $x=L$; see Figure 1). Along this extension it was assumed that both the upper and lower walls are impermeable and there is no slip. Hence, after a sufficiently long extension, gradients of the flow parameters with respect to x vanish. Then, according to this discussion, at the end of the computational domain (at $x=L+Le$) the following conditions can be imposed:

$$\left.\frac{\partial u}{\partial x}\right|_{x=L+Le} = 0. \tag{9a}$$

From equations (5), (9a) and (7b) we can write

$$v(L+Le,\ y) = 0. \tag{9b}$$

Concentration equation

It is assumed that the solute concentration is uniform at the channel inlet. As the solution flows down the channel, solvent is removed through the membrane. Hence a solute concentration profile develops and the solute concentration at the membrane surface increases. Assuming that the solute flows with the solvent (i.e. no slip between solvent and solute), the solute concentration equation can be written as

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \tag{10}$$

where C denotes the concentration of the solute and D is the diffusion coefficient for the solute. The value of D is taken to be 0·003 $m^2\ s^{-1}$. This equation is subject to the following boundary conditions.

At the inlet ($x = 0$).

$$C(0,\ y) = C_o. \tag{11}$$

On the upper wall ($y = h$).

$$\left.\frac{\partial C}{\partial y}\right|_{y=h} = 0. \tag{12}$$

At the bottom wall ($y = 0$). Under steady state conditions the particle accumulation near the porous boundary is determined by the balance between convective and diffusive transport, i.e.

$$D\left.\frac{\partial C}{\partial y}\right|_{y=0} = v_w C_w, \quad 0 \leqslant x \leqslant L. \tag{13a}$$

This boundary condition is valid for a perfectly rejecting membrane, i.e. no solid passes through the porous wall. In equation (13a), C_w denotes the unknown solute concentration on the porous wall. On the other hand, over the extended portion of the bottom wall the following condition is applied for the solute concentration:

$$\left.\frac{\partial C}{\partial y}\right|_{y=0} = 0, \quad L < x \leqslant L + Le. \tag{13b}$$

At the exit. Since the computational domain is extended beyond the end of the porous wall, at the right-hand boundary of the domain, the variation in the concentration with respect to x vanishes, i.e.

$$\left.\frac{\partial C}{\partial x}\right|_{x=L+Le} = 0. \tag{14}$$

3. NUMERICAL SOLUTION PROCEDURE

To transform the differential equations into algebraic equations, they were integrated over finite control volumes. The control volumes for the velocity components were staggered relative to those of the scalar variables. An upwind scheme was employed to approximate the convection and diffusion terms. Using the SIMPLE[6] algorithm, a computer programme was developed. For the solution of the algebraic equations the Gauss–Siedel point-by-point iteration technique was used. For all the grid points, u, v and P were solved. After the converged velocity field had been obtained, the concentration equation was solved to predict the solute distribution. Computations were terminated when the continuity for each control volume was satisfied within a magnitude of error less than 10^{-3}. In order to obtain grid-independent solutions, the computations were carried out with various grid sizes. It was found that a grid size of 132×14 (x, y) was reasonable from the viewpoints of accuracy and computer economy.

4. RESULTS AND DISCUSSION

The flow of viscous fluid between parallel plates of which the bottom one is permeable was analysed numerically. Velocity and concentration distributions were predicted over the flow field. The effects of the slip between the fluid and the porous wall on the velocity and particle concentration fields were investigated. The effects of the variation in the suction rate along the porous wall on the velocity and concentration fields were also studied. The computations were carried out for an inlet Reynolds number of 1000 and various wall (suction) Reynolds numbers.

As mentioned above, as the solvent flows through the porous wall, it also flows in the tangential direction over it. Hence a slip is present between the porous wall and the fluid. This slip affects the velocity and hence the concentration distribution in the flow field.

In order to investigate the effects of the slip on the flow field, the computations were performed with three different slip coefficients ($\phi_{slip} = 0{\cdot}0$, $0{\cdot}1$ and $0{\cdot}3$). Two sets of computations were performed. First, keeping the suction Reynolds number constant at 0·1, the slip coefficient was changed. Second, the wall Reynolds number (suction velocity) was varied linearly along the porous wall, being 0·15 at the inlet and 0·05 at the end of the porous wall.

Axial velocities were normalized by the local centreline velocity. In order to assess the accuracy of the present results, the velocity profile on the vertical mid-plane for a slip coefficient of 0·1 is compared with the results of Chellam *et al.*[3] in Figure 2. As can be seen, the data obtained in the present study match well with the results of Chellam *et al.*[3]

To see the effect of the slip on the velocity field, normalized axial velocities for three different slip coefficients are plotted in Figure 3. As the value of the slip coefficient increases, the wall shear stress decreases and hence the velocity on the porous wall increases. The maximum velocity moves towards the porous wall and the velocity profile becomes flatter with increasing slip. Dimensionless axial velocity profiles on the vertical mid-plane (at $x = L/2$) for linearly varying wall velocity are shown in Figure 4. When Figures 3 and 4 are compared, no significant differences between the corresponding profiles are seen. Since the suction rate in tangential membrane filtration systems is very small compared with the axial flow rate, a change in the suction velocity does not affect the behaviour of the flow to any considerable extent. However, it should be emphasized that the slip plays an important role in the flow behaviour.

In tangential membrane filtration systems the purpose is to separate particles from the solvent. The concentration of the particles is desired to be controlled during the filtration process. This requires control over the parameters that affect the concentration distribution in the flow field. In order to investigate the effects of the slip on the particle concentration distribution, the diffusion equation is

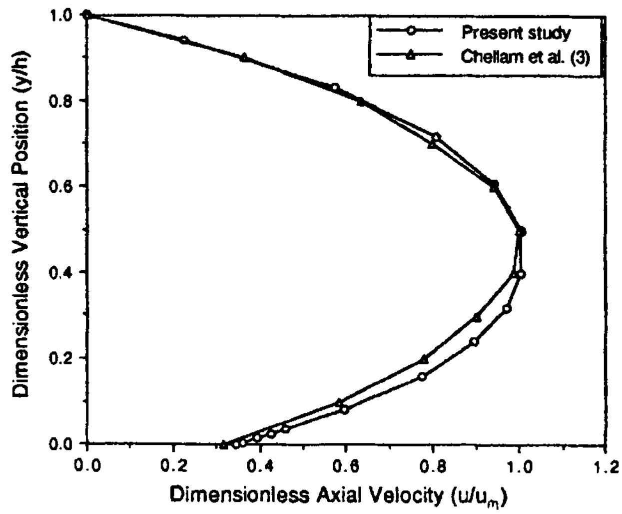

Figure 2. Comparison of velocity profile on vertical mid-plane with results of Chellam *et al.*[3]

solved for three different slip coefficients, keeping the other flow parameters constant ($Re_{in} = 1000$, $Re_w = 0{\cdot}1$). Dimensionless concentration (C/C_o) profiles on the vertical mid-plane are shown in Figure 5. When this figure is analysed, it is seen that concentration increases towards the porous wall and reaches its maximum value on the wall. It should be noted that a variation in the concentration exists only near the permeable wall. It is also seen that the concentration near the wall decreases with increasing slip. This could be attributed to the fact that the flow rate in the streamwise direction near the wall increases with increasing slip, while the suction rate stays the same.

During the filtration processes, as the solvent flows through the porous wall, the solute concentration over the porous surface increases in the flow direction. This increase in the particle concentration on the porous wall reduces the porosity of the wall. In addition, the pressure decreases along the channel.

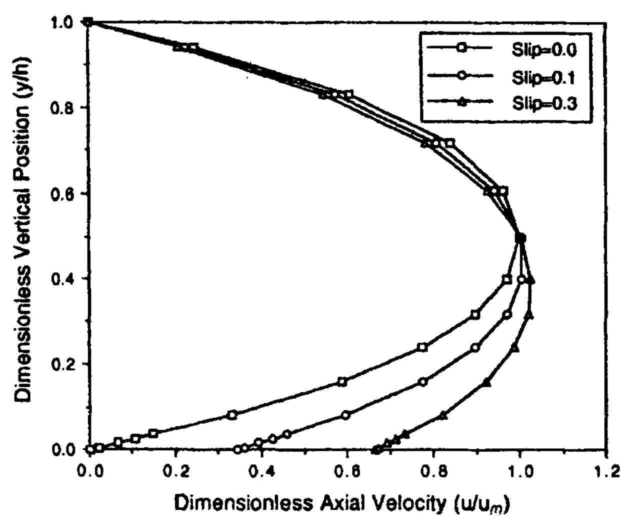

Figure 3. Profiles of x-component of velocity on vertical mid-plane for constant transverse wall velocity cases

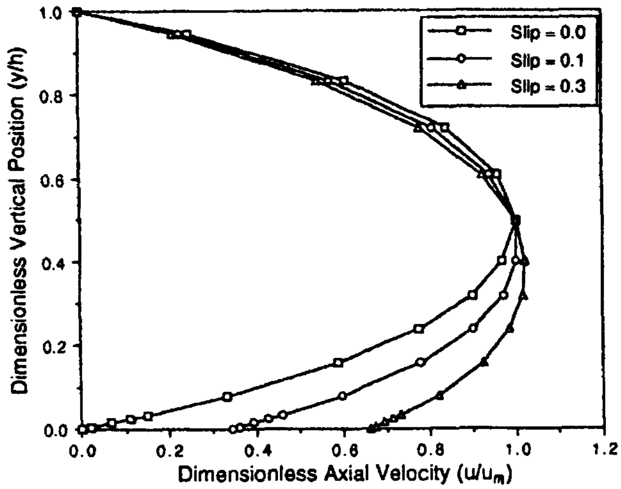

Figure 4. Profiles of x-component of velocity on vertical mid-plane for variable transverse wall velocity cases

As a result, the rate of solvent flow through the porous wall decreases in the flow direction. In order to simulate this physical situation, the variation in the suction rate along the porous wall should be taken into account in the analysis. The variations in the concentration along the channel on the porous wall for variable transverse velocity ($v_w = -0{\cdot}15 + 0{\cdot}10x/L$) and constant transverse velocity ($v_w = -0{\cdot}10$) are shown in Figure 6. For the constant transverse velocity case the concentration on the porous wall increases with the distance. However, for the variable suction velocity case the concentration profile reaches a maximum value. The concentration profile obtained when the suction rate is constant is not physically realistic because of the reasons expressed above. Therefore in the analysis of the problem the variation in the suction rate along the channel should be taken into account.

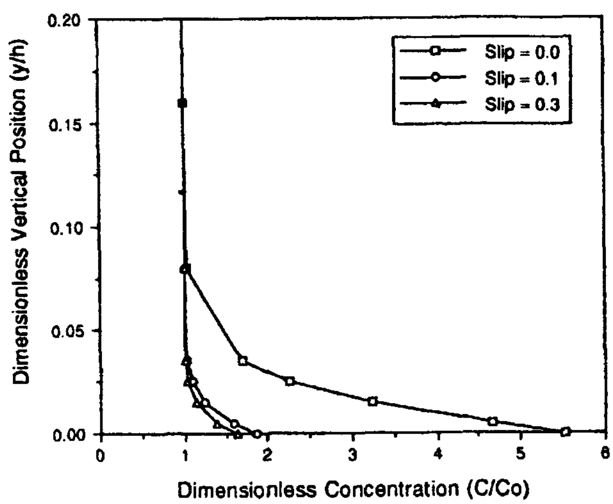

Figure 5. Concentration profiles along vertical mid-plane for variable transverse wall velocity cases

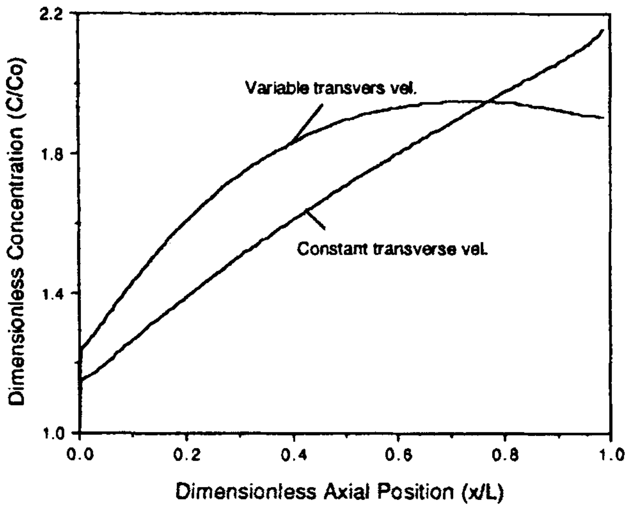

Figure 6. Variation in concentration on porous wall in flow direction (slip = 0·1)

5. CONCLUSIONS

The flow of solutions between parallel plates of which the bottom one is porous was analysed numerically. The velocity and concentration distributions were predicted. The effects of the slip between the fluid and the porous wall on the velocity and particle concentration distributions were investigated. As the slip increases, the concentration on the porous wall decreases and the velocity profile becomes flatter. It was also observed that the concentration on the porous wall increases along the flow when the transverse velocity through the porous wall is constant. This accumulation of particles on the porous wall causes a reduction in the porosity of the wall and hence a reduction in the suction rate. In the present study this situation was taken into account by using a linearly varying transverse velocity along the porous wall and a concentration profile which is different from the constant transverse velocity case was observed. A linearly varying suction rate boundary condition resulted in a physically realistic concentration profile along the wall.

APPENDIX: NOMENCLATURE

C solute concentration
C_o solute concentration at inlet
C_w solute concentration on porous wall surface
D solute diffusivity
h height of channel
k membrane permeability
L length of porous wall
Le length of extended computational domain
P pressure
Re_{in} inlet Reynolds number ($Re_{in} = hU_o/v$)
Re_w wall Reynolds number ($Re_w = hv_w/v$)
u velocity component in x-direction
u_m x-component of velocity on centreline

u_o velocity at inlet
v velocity component in y-direction
v_w transverse velocity on porous wall
x axial distance from channel entrance
y vertical distance from porous wall

Greek letters

α dimensionless constant depending on surface characteristic of membrane
ν kinematic viscosity
ρ density
ϕ_{slip} slip coefficient ($\phi_{slip} = \sqrt{k}/\alpha$)

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